Dirichlet Series II

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Lemma 1

Let $\{f_n\}$ be a sequence of functions analytic on an open subset S of the complex plane, and assume that $\{f_n\}$ converges uniformly on every compact subset of S to a limit function f. Then f is analytic on S and the sequence of derivatives $\{f_n'\}$ converges uniformly on every compact subset of S to the derivative f'.

Sketch of the proof.

Since f_n is analytic on S, we have Cauchy's integral formula:

$$f_n(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(z)}{z-a}\, dz,$$

where D is any compact disk in S. Since $\{f_n\}$ converges uniformly on every compact subset of S to a limit function f, we have

$$f(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-a}\, dz.$$

Hence we have the following derivatives

$$f_n'(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_n(z)}{(z-a)^2}) \, dz,$$

$$f'(a) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-a)^2} \, dz.$$

Teorem 1

A Dirichlet series $\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ *converges uniformly on every compact subset lying interior to the half-plane of convergence* $\sigma > \sigma_c$.

Sketch of the proof. Let R be a compact rectangle such that

$$R = \{s = \sigma + it \,:\, \alpha \leq \sigma \leq \beta \text{ and } \gamma \leq t \leq \delta\} \subset \{\Re s > \sigma_c\}.$$

From the estimate showed in the last week seminar Dirichlet Series I, we have

$$\left| \sum_{a<n\leq b} f(n)\, n^{-s} \right| \leq 2M\, a^{\sigma_0 - \sigma} \left(1 + \frac{|s - s_0|}{\sigma - \sigma_0}\right).$$

Teorem 2

The sum function $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ *of a Dirichlet series is analytic in its half-plane of convergence* $\sigma > \sigma_c$, *and its derivative* $F'(s)$ *is represented in this half-plane by the Dirichlet series*

$$F'(s) = -\sum_{n=1}^{\infty} \frac{f(n)\log n}{n^s}, \tag{11}$$

obtained by differentiating term by term.

Applying this theorem repeatedly yields

$$F^{(k)}(s) = (-1)^k \sum_{n=1}^{\infty} \frac{f(n)\,(\log n)^k}{n^s} \text{ for } \sigma > \sigma_c.$$

Teorem 3

Let $F(s)$ be represented in the half-plane $\sigma > c$ by the Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s},$$

where c is finite, and assume that $f(n) \geq 0$ for all $n \geq n_0$. If $F(s)$ is analytic in some disk about the point $s = c$, then the Dirichlet series converges in the half-plane $\sigma > c - \varepsilon$ for some $\varepsilon > 0$. Consequently, if the Dirichlet series has a finite abscissa of convergence σ_c, then $F(s)$ has a singularity on the real axis at the point $s = \sigma_c$.

Sketch of the proof. Let $a = 1 + c$. By assumption, F is analytic at $s = c$, so F can be represented by an absolutely convergent power series expansion about a:

$$F(s) = \sum_{k=0}^{\infty} \frac{F^{(k)}(a)}{k!} (s-a)^k,$$

and the radius of convergence exceeds 1. Then

$$F(s) = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} \frac{(a-s)^k}{k!} \, f(n) \, (\log n)^k \, n^{-a}.$$

This formula is valid for some real $s = c - \epsilon$ where $\epsilon > 0$.

$$\begin{aligned} F(c-\varepsilon) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^a} \sum_{k=0}^{\infty} \frac{\{(1+\varepsilon)\log n\}^k}{k!} \\ &= \sum_{n=1}^{\infty} \frac{f(n)}{n^a} \, e^{(1+\varepsilon)\log n} \\ &= \sum_{n=1}^{\infty} \frac{f(n)}{n^{\,c-\varepsilon}}. \end{aligned}$$

Teorem 4

Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ *be absolutely convergent for* $\sigma > \sigma_a$ *and assume that* $f(1) \neq 0$*. If* $F(s) \neq 0$ *for* $\sigma > \sigma_0 \geq \sigma_a$*, then for* $\sigma > \sigma_0$ *we have*

$$F(s) = e^{G(s)}$$

with

$$G(s) = \log f(1) + \sum_{n=2}^{\infty} \frac{(f' * f^{-1})(n)}{n^s \log n},$$

where f^{-1} *is the Dirichlet inverse of* f *and* $f'(n) = f(n) \log n$*.*

Teorem 5

Given two Dirichlet series $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ *and* $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ *with abscissae of absolute convergence* σ_1 *and* σ_2*, respectively. Then for* $a > \sigma_1$ *and* $b > \sigma_2$ *we have*

$$\frac{1}{2T}\int_{-T}^{T} F(a+it)\,G(b-it)\,dt = \sum_{n=1}^{\infty} \frac{f(n)g(n)}{n^{a+b}}.$$

Sketch of the proof. We have

$$\begin{aligned} F(a+it)\,G(b-it) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{f(m)g(n)}{m^a n^b}\left(\frac{n}{m}\right)^{it} \\ &= \sum_{n=1}^{\infty} \frac{f(n)g(n)}{n^{a+b}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{f(m)g(n)}{m^a n^b}\left(\frac{n}{m}\right)^{it}. \end{aligned}$$

$$\frac{1}{2T}\int_{-T}^{T} F(a+it)\,G(b-it)\,dt$$

$$= \sum_{n=1}^{\infty} \frac{f(n)g(n)}{n^{\,a+b}} \;+\; \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{f(m)g(n)}{m^a n^b}\,\frac{1}{2T}\int_{-T}^{T} e^{\,it\,\log(n/m)}\,dt$$

$$= \sum_{n=1}^{\infty} \frac{f(n)g(n)}{n^{\,a+b}} \;+\; \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{f(m)g(n)}{m^a n^b}\,\frac{\sin\big(T\log(n/m)\big)}{T\,\log(n/m)}\,.$$

Teorem 6

If $F(s) = \sum_{n=1}^{\infty} f(n) n^{-s}$ *converges absolutely for* $\sigma > \sigma_a$*, then for* $\sigma > \sigma_a$ *we have*

$$\frac{1}{2T} \int_{-T}^{T} \left| F(\sigma + it) \right|^2 dt = \sum_{n=1}^{\infty} \frac{|f(n)|^2}{n^{2\sigma}}.$$

Teorem 6, cont.

In particular, if $\sigma > 1$ we have

$$\textit{(a)} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|\zeta(\sigma+it)\right|^2 dt = \sum_{n=1}^{\infty} \frac{1}{n^{2\sigma}} = \zeta(2\sigma),$$

$$\textit{(b)} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|\zeta^{(k)}(\sigma+it)\right|^2 dt = \sum_{n=1}^{\infty} \frac{\log^{2k} n}{n^{2\sigma}} = \zeta^{(2k)}(2\sigma)\ ,$$

$$\textit{(c)} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|\zeta(\sigma+it)\right|^{-2} dt = \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^{2\sigma}} = \frac{\zeta(2\sigma)}{\zeta(4\sigma)},$$

$$\textit{(d)} \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \left|\zeta(\sigma+it)\right|^4 dt = \sum_{n=1}^{\infty} \frac{\sigma_0^2(n)}{n^{2\sigma}} = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}.$$

Teorem 7

Assume the series $F(s)=\sum_{n=1}^{\infty} f(n)\, n^{-s}$ *converges absolutely for* $\sigma>\sigma_a$. *Then for* $\sigma>\sigma_a$ *and* $x>0$ *we have*

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} F(\sigma+it)\, x^{\sigma+it}\, dt=\begin{cases} f(n), & \text{if } x=n,\\ 0, & \text{otherwise.}\end{cases}$$

Sketch of the proof. For $\sigma>\sigma_a$,

$$\frac{1}{2T}\int_{-T}^{T} F(\sigma+it)\, x^{\sigma+it}\, dt=\frac{x^{\sigma}}{2T}\int_{-T}^{T}\sum_{n=1}^{\infty}\frac{f(n)}{n^{\sigma}}\left(\frac{x}{n}\right)^{it} dt$$

$$=\frac{x^{\sigma}}{2T}\sum_{n=1}^{\infty}\frac{f(n)}{n^{\sigma}}\int_{-T}^{T} e^{\,it\,\log(x/n)}\, dt\,.$$

Lemma 2

If $c > 0$*, define* $\int_{c-i\infty}^{c+i\infty}$ *to mean* $\lim_{T\to\infty}\int_{c-iT}^{c+iT}$*. Then if* a *is any positive real number, we have*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{a^z}{z}\,dz = \begin{cases} 1, & a > 1, \\ \frac{1}{2}, & a = 1, \\ 0, & 0 < a < 1. \end{cases}$$

Lemma 2,cont.

Moreover, we have

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{a^z}{z}\,dz\right| < \frac{2a^c}{T\log\frac{1}{a}} \quad (0<a<1),$$

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{a^z}{z}\,dz - 1\right| < \frac{a^c}{\pi T\log a} \quad (a>1),$$

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{a^z}{z}\,dz - \frac{1}{2}\right| < \frac{c}{\pi T} \quad (a=1).$$

Teorem 8

Let $F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$ *be absolutely convergent for* $\sigma > \sigma_a$*; let* $c > 0$*,* $x > 0$ *be arbitrary. Then if* $\sigma > \sigma_a - c$ *we have*

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s+z) \frac{x^z}{z} \, dz = \sum_{n \le x}^{*} \frac{f(n)}{n^s},$$

where $\sum^*$ *means that the last term in the sum must be multiplied by* $1/2$ *when* x *is an integer.*

References

- T. M. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976